Leveraged ETF options' implied volatility paradox: a statistical study

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Implied & Local Volatility

- Implied Volatility (IV): $\widehat{\sigma} : (t, K, T) \to \widehat{\sigma}_t(K, T)$: the BS option price implied measure of volatility
- local volatility (LV): under risk-neutral measure Q assume

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t, \cdot)dW_t^{\mathrm{Q}}, \tag{1}$$

then the local volatility is given by

$$\sigma_{K,T}(S_t, t) \stackrel{\text{def}}{=} \sqrt{\sigma^2_{K,T}(S_t, t)} = \sqrt{\mathrm{E}^{\mathrm{Q}}\{\sigma^2(S_T, T, \cdot)|S_T = K, \mathcal{F}_t\}} \tag{2}$$

with $(\Omega, \mathcal{F}, P)$ probability space, $\{\mathcal{F}_t\}_{0 \le t \le T}$ filtration

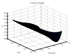
(L)ETFs

- Exchange-traded funds (ETFs): tracking returns on financial quantities and yielding the identical daily return,
  e.g., SPDR S&P 500 ETF (SPY) tracks the S&P 500.

- Leveraged exchange-traded funds (LETFs): promising a fixed leverage ratio β w.r.t. a given underlying asset or index,
  e.g.,

LETF	β
ProShares Ultra S&P500 (SSO)	2
ProShares UltraPro S&P 500 (UPRO)	3
ProShares UltraShort S&P500 (SDS)	-2
ProShares UltraPro Short S&P 500 (SPXU)	-3

Table 1: LETFs with different β ▸ Illustration



Go with market

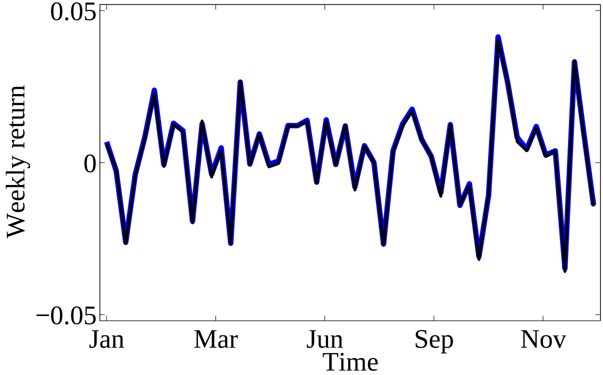


Figure 1: Weekly returns of ETF (SPY) and stock market (S&P 500) (20140101-20141230)

Leverage up/down

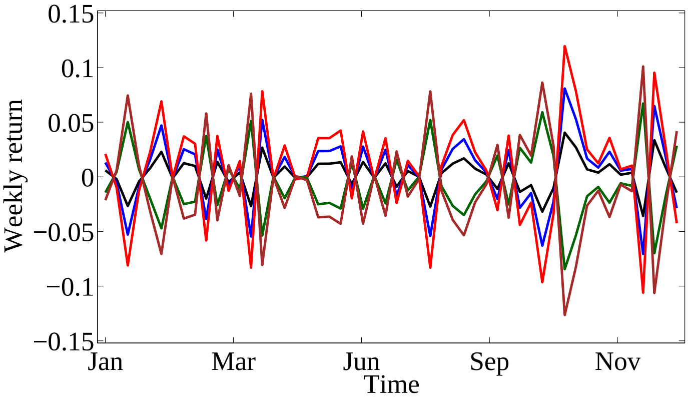


Figure 2: Weekly returns of LETFs (SSO, UPRO, SDS, SPXU) and stock market (S&P 500) (20140101-20141230)

Implied volatility paradox

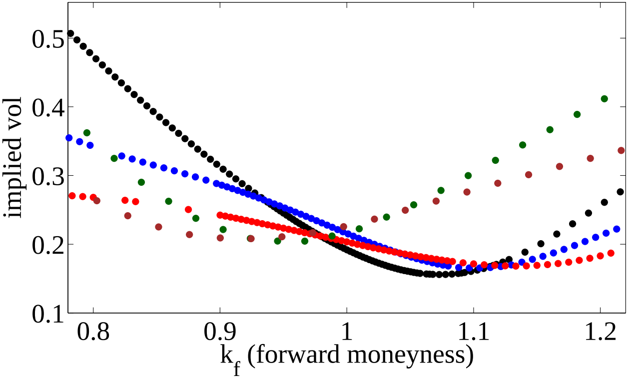


Figure 3: Implied volatility of (L)ETF options (SPY, SSO, UPRO, SDS, SPXU) with 21 days to maturity

Objectives

- introduce moneyness scaling technique
- study statistical significance of moneyness scaling
- identify LETF option price discrepancies using moneyness scaling
- introduce a dynamic model for IVS
- build a trading strategy based on possible arbitrage
- extend the model to a stochastic case

Outline

1. Motivation ✓
2. Moneyness scaling
3. Confidence bands
4. Dynamic estimation
5. Practical implications
6. Stochastic volatility
7. Conclusions

LETFs and the Black-Scholes model

- asset price dynamics:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{Q} \tag{3}$$

with interest rate r and volatility σ; W_t^{Q} standard Brownian motion under the risk-neutral measure P^{Q}

- (L)ETF dynamics:

$$\begin{aligned} \frac{dL_t}{L_t} &= \beta\left(\frac{dS_t}{S_t}\right) - \{(\beta-1)r + c\}dt \\ &= (r-c)dt + \beta\sigma dW_t^{Q} \end{aligned} \tag{4}$$

$0 \leq c \ll r$ (L)ETF expense ratio

Moneyness scaling (MS)

- with forward moneyness measure $\kappa_f \stackrel{\text{def}}{=} K/\{e^{(r-c)\tau} L_t\}$ and time to maturity τ:

$$\kappa_f{}^{(\beta_1)} = \exp\left\{-\frac{\beta_1}{2}(\beta_1-\beta_2)\overline{\sigma}^2\tau\right\}\left(\kappa_f{}^{(\beta_2)}\right)^{\frac{\beta_1}{\beta_2}}, \tag{5}$$

where $\overline{\sigma}$ is the average IV across all strikes Details

- with log-moneyness measure $LM \stackrel{\text{def}}{=} \log(K/L_t)$, Leung and Sircar (2015)

$$LM^{\beta_1} = \frac{\beta_1}{\beta_2}\left[LM^{\beta_2} + \{r(\beta_2-1)+c_2\}\tau + \frac{\beta_2(\beta_2-1)}{2}\overline{\sigma}^2\tau\right] - \{r(\beta_1-1)+c_1\}\tau - \frac{\beta_1(\beta_1-1)}{2}\overline{\sigma}^2\tau \tag{6}$$

Data example

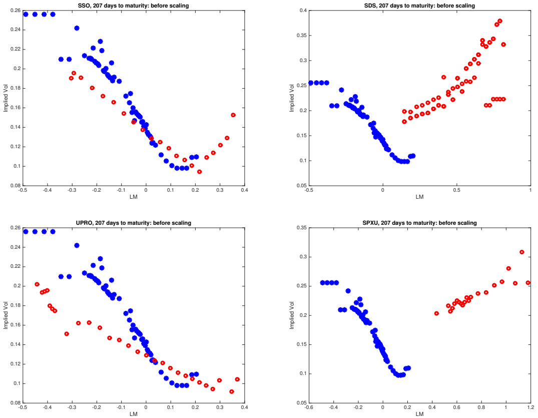


Figure 4: SPY and LETFs implied volatilities before scaling on June 23, 2015 with 207 days to maturity, plotted against their log-moneyness

Data example

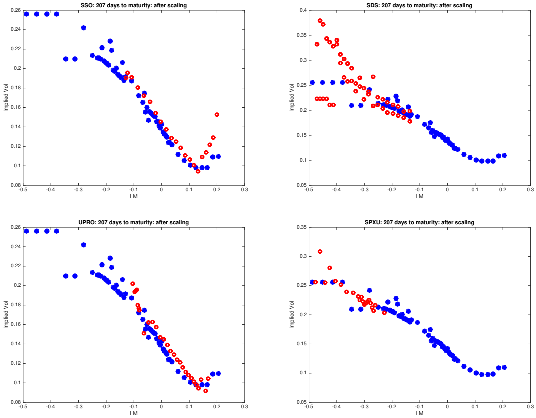


Figure 5: SPY and LETFs implied volatilities after moneyness scaling on June 23, 2015 with 207 days to maturity, plotted against their log-moneyness

Sustainability of MS effect

- is moneyness scaling effect statistically significant?
- how can one study MS sustainability?
- are there practical implications?
- extension to stochastic volatility

Confidence bands

1 Compute the estimate $\widehat{m}_h(X)$ by a local linear M-smoothing procedure with some kernel function and bandwidth h chosen by, e.g., cross-validation.

2 Given $\widehat{\varepsilon}_t \stackrel{\text{def}}{=} Y_t - \widehat{m}(X_t)$; $\widehat{m}$ from Step 1, do a bootstrap resampling from $\widehat{\varepsilon}_t$, that is, for each $t = 1, \ldots, T$, generate a random variable $\varepsilon_i^* \sim \widehat{F}_{\varepsilon|X}(z)$ and a re-sample

$$Y_t^* = \widehat{m}_g(X_t) + \varepsilon_t^*, \quad t = 1, \ldots, T \tag{7}$$

B times (bootstrap replications). with an "oversmoothing" bandwidth $g \gg h$ such as $g = \mathcal{O}(T^{-1/9})$.

Confidence bands

3 For each re-sample $\{X_t, Y_t^*\}_{t=1}^T$ compute $\widehat{m}_{h,g}^*$ using the bandwidth h and construct the random variable

$$d_b \stackrel{\text{def}}{=} \sup_{x \in B} \left[\frac{|\widehat{m}_{h,g}^*(x) - \widehat{m}_g(x)| \sqrt{\widehat{f}_X(x) \widehat{f}_{\varepsilon|X=x_t}(\varepsilon_t^*)}}{\sqrt{\widehat{\mathsf{E}}_{Y|X}\{\psi^2(\varepsilon_t^*)\}}} \right], \qquad (8)$$

where $b = 1, \dots, B$, $\psi(u) = \rho'(\cdot)$, ρ robust loss function

4 Calculate the $1 - \alpha$ quantile d_α^* of $d_1, \dots, d_B$.

5 Construct

$$\widehat{m}_h(x) \pm \left[\frac{\sqrt{\widehat{\mathsf{E}}_{Y|X}\{\psi^2(\varepsilon_t^*)\}} d_\alpha^*}{\sqrt{\widehat{f}_X(x) \widehat{f}_{\varepsilon|X=x_t}(\varepsilon_t^*)}} \right] \qquad (9)$$

Data analysis

- daily data in 20141117-20151117 for the LETFs SSO, UPRO, SDS
- M-smoother estimator of implied volatility Y given forward moneyness X
- times-to-maturity: 0.5, 0.6, 0.7 years
- $B = 1000$

Data analysis

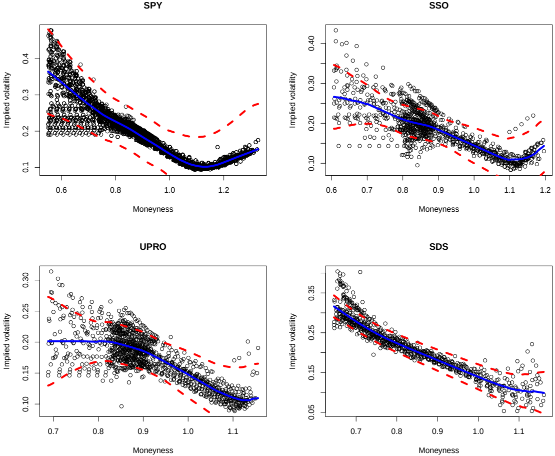


Figure 6: Fitted implied volatility and bootstrap uniform confidence bands for 4 (L)ETFs on S&P500; τ: 0.6 years

Data analysis

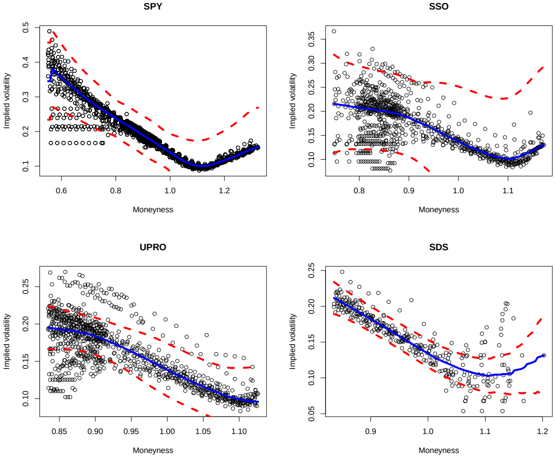


Figure 7: Fitted implied volatility and bootstrap uniform confidence bands for 4 (L)ETFs on S&P500; τ: 0.5 years

Data analysis

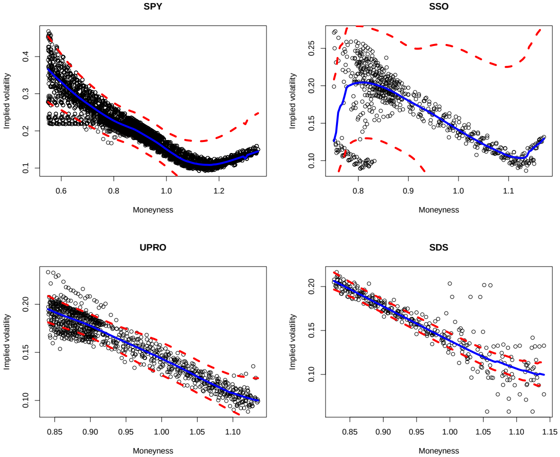


Figure 8: Fitted implied volatility and bootstrap uniform confidence bands for 4 (L)ETFs on S&P500; τ: 0.7 years

Combined bands

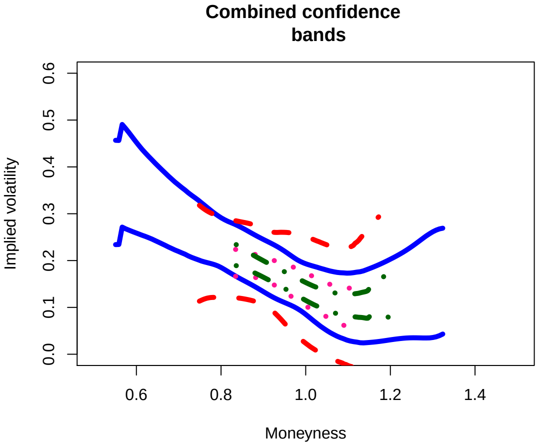


Figure 9: Combined uniform bootstrap confidence bands for SPY, SSO, UPRO and SDS after MS; τ: 0.5 years

Combined bands

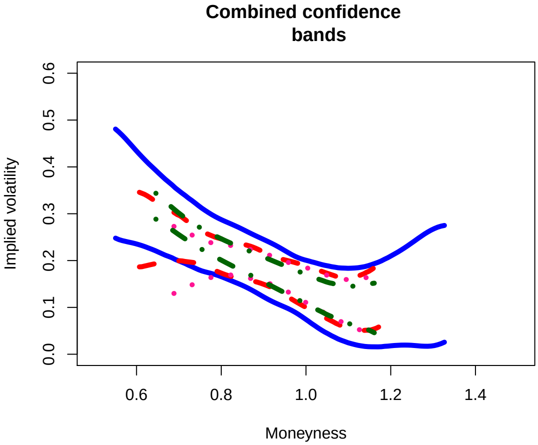


Figure 10: Combined uniform bootstrap confidence bands for SPY, SSO, UPRO and SDS after MS; τ: 0.6 years

Combined bands

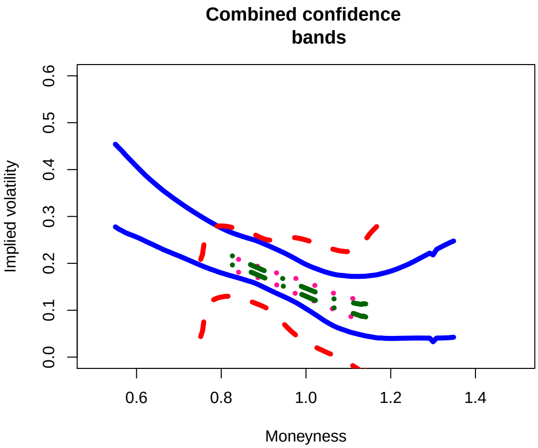


Figure 11: Combined uniform bootstrap confidence bands for SPY, SSO, UPRO and SDS after MS; τ: 0.7 years

Challenges for IV estimation

- how to model the IV surface?
- degenerated data design: IVS observations only for a small number of maturities
- observation grid does not cover desired estimation grid
  - the contracts are not traded for a particular strike
  - institutional arrangements at the futures' exchanges

Implied volatility in time

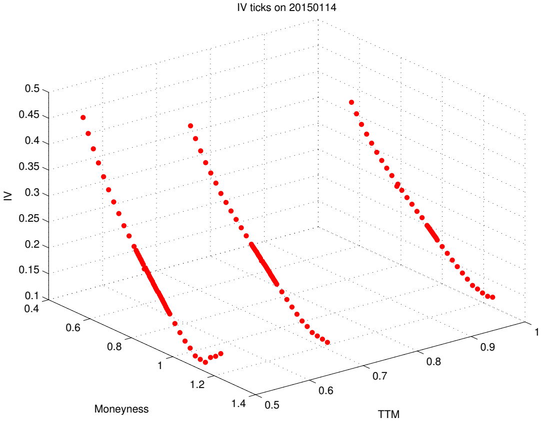


Figure 12: SPY ETF option IV ticks of 20150114-20150408

IV data design

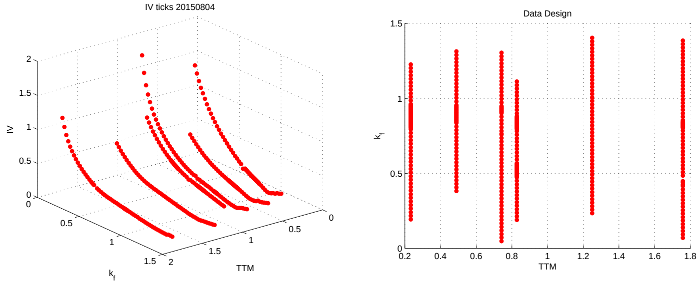


Figure 13: *Left panel*: SPY call IV observed on 20150408; *Right panel*: data design on 20150408

The dynamic semiparametric factor model for IVS

- define $\mathcal{J} \stackrel{\text{def}}{=} [\kappa_{min}, \kappa_{max}] \times [\tau_{min}, \tau_{max}]$, $Y_{i,j}$ implied volatility, $i = 1, \ldots, I$ time index, $j = 1, \ldots, J_i$ option intraday numbering on day i, $X_{i,j} \stackrel{\text{def}}{=} (\kappa_{i,j}, \tau_{i,j})^\top$, $Y_{i,j} \stackrel{\text{def}}{=} \sigma_{i,j}$ implied volatility; κ some measure of moneyness, e.g., log- or forward
- assume

$$Y_{i,j} = \mathcal{Z}_i^\top m(X_{i,j}) + \varepsilon_{i,j}, \tag{10}$$

  where $\mathcal{Z}_i = (1, Z_i^\top)$, $Z_i = (Z_{i,1}, \ldots, Z_{i,L})^\top$ unobservable L-dimensional process, $m = (m_0, \ldots, m_L)^\top$, real-valued functions m_l, $l = 1, \ldots, L+1$ defined on a subset of $\mathbb{R}^d$
- $X_{i,j}$, $\varepsilon_{i,j}$ are independent, $\varepsilon_{i,j} \sim (0, \sigma^2)$, $\sigma^2 < \infty$

DSFM for IVS

Approximate, Park et al. (2009):

$$\mathsf{E}(Y_i|X_i) = \mathcal{Z}_i^\top m(X_i) = \mathcal{Z}_i^\top \mathcal{A}\psi(X_i), \tag{11}$$

where

$$\psi(X_i) \stackrel{\text{def}}{=} \{\psi_1(X_i), \ldots, \psi_K(X_i)\}^\top \text{ space basis,}$$
$$\mathcal{A} : (L+1) \times K \text{ coefficient matrix}$$

Choose $\{\psi_k : 1 \le k \le K\}$ tensor B-spline basis Details, de Boor (2001)

K is the number of tensor B-spline sites s_k^τ, s_k^κ

Tensor B-spline basis

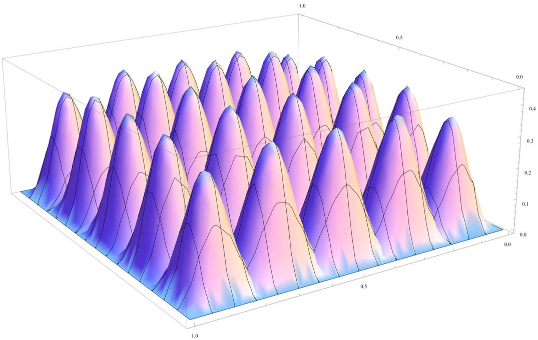


Figure 14: Tensor B-spline basis with 15×15 knots on $[0, 1] \times [0, 1]$, odd intervals

Estimation

Define the least-squares estimators $\widehat{Z}_i = (\widehat{Z}_{i,1}, \ldots, \widehat{Z}_{i,L})^\top$,
$\widehat{\mathcal{A}} = (\widehat{\alpha}_{l,k})_{l=0,\ldots,L;k=1,\ldots,K}$

$$(\widehat{Z}_i, \widehat{\mathcal{A}}) = \arg\min_{Z_i, \mathcal{A}} S(\mathcal{A}, Z), \tag{12}$$

where

$$S(\mathcal{A}, Z) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \left\{ Y_{i,j} - (1, Z_i^\top)\mathcal{A}\psi(X_{i,j}) \right\}^2 \tag{13}$$

Once $\widehat{\mathcal{A}}$ obtained, m can be estimated as $\hat{m} = \widehat{\mathcal{A}}\psi$

Identification

- the problem (12) can be solved via numeric algorithm ▸ Details
- under certain conditions ▸ Details, geometric convergence to a solution
- (12) has no unique solution: orthonormalize $\widehat{Z}_i$, $\hat{m}$ for better interpretation, see Fengler et al. (2007)

Why DSFM?

- can model and forecast the whole IV surface:

$$\widehat{IV}_{t;i,j} = \widehat{m}_{0;i,j} + \sum_{l=1}^{L} \widehat{\mathcal{Z}}_{l,t} \widehat{m}_{l;i,j}, \tag{14}$$

  where

$$\widehat{m}_{l;i,j} = \sum_{i}^{|s^\kappa|} \sum_{j}^{|s^\tau|} \widehat{A}_{l;i,j} \psi_{i,k_\kappa}(\kappa_i) \psi_{j,k_\tau}(\tau_j), \tag{15}$$

  k_κ, k_τ knot sequences, s^κ, s^τ site sets; $\widehat{\mathcal{A}}$ reshaped into a $|s^\kappa| \times |s^\tau| \times L$ array of L matrices $\widehat{A}$ of dimension $|s^\kappa| \times |s^\tau|$

- stochastic loadings $\widehat{\mathcal{Z}}_t$ have vector autoregressive (VAR) dynamics

Data overview

		Min.	Max.	Mean	Stdd.	Skewn.	Kurt.
SPY	TTM	0.26	1.05	0.76	0.19	−0.54	2.76
	Moneyness	0.05	1.43	0.48	0.17	−0.34	3.15
	IV	0.25	1.55	0.46	0.23	1.94	7.17
SSO	TTM	0.21	1.04	0.63	0.25	0.01	1.76
	Moneyness	0.18	1.69	0.63	0.29	0.92	3.61
	IV	0.25	1.34	0.41	0.11	1.91	10.81

Table 2: Summary statistics on SPY, SSO ETF options from 20140920 to 20150630 (in total $\sum_i J_t = 9828, 7619$ datapoints, respectively). Source: Datastream

Estimation

- estimation space $[\kappa_{min}, \kappa_{max}] \times [\tau_{min}, \tau_{max}]$ re-scaled (via marginal edf) to $[0,1]^2$
- 9 knots in moneyness and 7 knots in maturity direction, cubic splines ($k = 3$) in both directions, so $K = 24$
- starting values for Z_i generated from a stable VAR process
  Details
- starting values for $\mathcal{A}$ randomly generated from $U(0,1)$
- convergence tolerance for the Newton algorithm: 1e-06

Model order selection

Select model order by explained variance $EV(L)$

$$EV(L) \stackrel{\text{def}}{=} 1 - \frac{\sum_{i=1}^{I} \sum_{j=1}^{J_i} \left\{ Y_{i,j} - \sum_{l=0}^{L} \widehat{Z}_{i,l} \hat{m}_l(X_{i,j}) \right\}^2}{\sum_{i=1}^{I} \sum_{j=1}^{J_i} (Y_{i,j} - \overline{Y})^2} \tag{16}$$

Root-mean-squared-error (RMSE) measures goodness-of-fit

$$RMSE \stackrel{\text{def}}{=} \sqrt{\frac{1}{\sum_t J_t} \sum_{t=1}^{T} \sum_{j=1}^{J_t} \left\{ Y_{t,j} - \sum_{l=0}^{L} \widehat{Z}_{t,l} \widehat{m}_l(X_{t,j}) \right\}^2} \tag{17}$$

Model order selection

Criterion	$L = 2$	$L = 3$	$L = 4$	$L = 5$
$EV(L)$	0.915	0.921	0.925	0.930
$RMSE$	0.090	0.088	0.087	0.082

Table 3: *Explained variance and RMSE criteria for different model order sizes*

Estimate $L = 3$ basis functions

Dynamics of $\widehat{Z}_i$

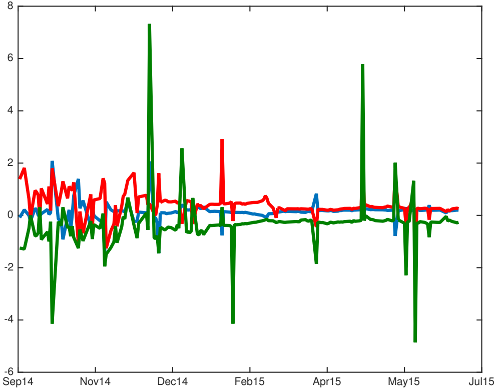


Figure 15: Time dynamics of $\widehat{Z}_{i,1}$, $\widehat{Z}_{i,2}$, $\widehat{Z}_{i,3}$

VAR modelling of $\widehat{Z}_i$

Model order n	AIC(n)	HQ(n)	SC(n)
1	-4.20^*	-4.10^*	-3.96^*
2	-4.13	-3.96	-3.72
3	-4.07	-3.83	-3.48
4	-4.03	-3.72	-3.27
5	-3.97	-3.59	-3.03

Table 4: The VAR model selection criteria. The smallest value is marked by an asterisk

VAR modelling of $\widehat{Z}_i$

- all roots of VAR(1) model lie inside the unit circle
- Portmanteau and Breusch-Godfrey LM test results with 12 lags fail to reject residual autocorrelation at 10% significance level

Estimated factor functions $\hat{m}_l$

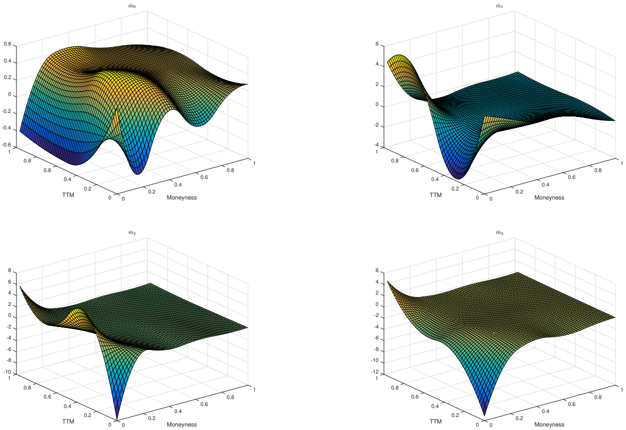


Figure 16: Factor functions $\hat{m}_0$, $\hat{m}_1$, $\hat{m}_2$, $\hat{m}_3$

Bias comparison

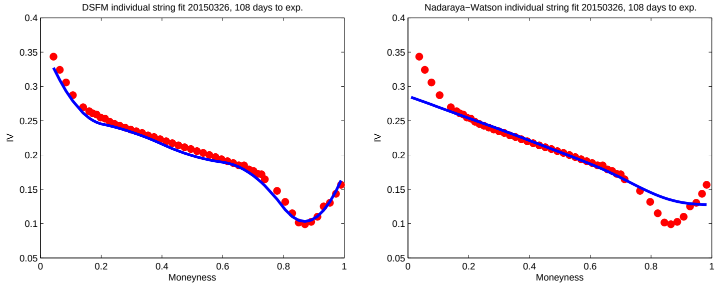


Figure 17: Bias comparison of the DSFM (*left panel*) and the Nadaraya-Watson estimator with $\hat{h} = (0.13, 0.12)^\top$, $\hat{h}$ by Scott's rule, for the 108 days to expiry data (red dots) on 20150326

Strategy motivation

- "trade-with-the-smile/skew" strategy adapted for the special case of ETF-LETF option IV discrepancy
- use the ETF option data to estimate the model (theoretical) smile of the leveraged counterpart and the information from the IV surface forecast to recognize the future (one-period-ahead) possible IV discrepancy
- delta-hedging is required

Strategy setup

Choose moving window width w; for each $t = w, \ldots, T$ do:

1. Given $\beta_2 = 1$, β_1, re-scale LM^{β_2} according to the MS formula (6) to obtain $\widehat{LM}^{\beta_1}$, obtain the "model" moneyness coordinate for DSFM estimation.
2. Estimate (10) on $[\widehat{LM}^{\beta_1}_{min}, \widehat{LM}^{\beta_1}_{max}] \times [\tau^{SPY}_{min}, \tau^{SPY}_{max}]$ (re-scaled to $[0,1]^2$), obtain the IV surface estimates $\widehat{IV}_1, \ldots, \widehat{IV}_t$.
3. Forecast $\widehat{IV}_{t+1}$ using the VAR structure of $\widehat{\mathcal{Z}}_t$ and factor functions $\widehat{m}_l$.

Strategy setup

4. Choose a specific IV "string" for some τ^* at time point t using SSO option data and calculate the marginally transformed $\widehat{LM}^{\beta_1}_{\tau^*}$ of the *true* SSO log-moneyness LM^{β_1} using the marginal distribution of $\widehat{LM}^{\beta_1}$.
5. Using $\widehat{LM}^{\beta_1}_{\tau^*}$, τ^* and $\widehat{IV}_{t+1}$, interpolate the "theoretical" IV $\widehat{IV}_{t+1}$ over the marginally re-scaled $[\tau^*, \tau^*] \times [\widehat{LM}^{\beta_1}_{min}, \widehat{LM}^{\beta_1}_{max}]$ to obtain "theoretical" values $\widehat{IV}_{t+1; LM^{\beta_1}_{\tau^*}, \tau^*}$.

Strategy setup

6. Compare "theoretical" $\widehat{IV}_{t+1;LM^{\beta_1}_{\tau^*},\tau^*}$ with "true" $IV_{t;LM^{\beta_1}_{\tau^*},\tau^*}$ and construct a delta-hedged option portfolio: buy (long) an option with the absolute largest negative deviation from the "theoretical" IV (IV expected to fall) and sell (short) an option with the smallest positive deviation from the "theoretical" IV (IV expected to increase). Use the underlying SSO LETF asset to make the whole portfolio delta-neutral.

7. At time point $t+1$, terminate the portfolio, calculate profit/loss and repeat until time T.

Discrepancy 1

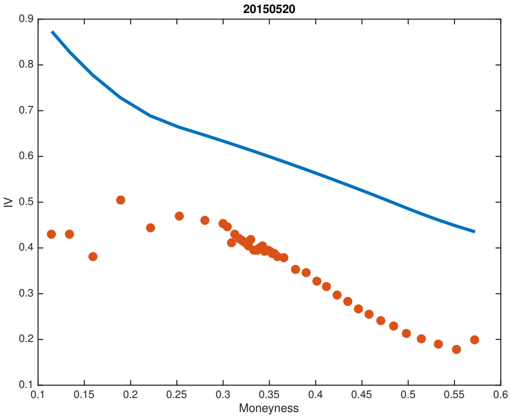


Figure 18: True SSO and MS-DSFM-modeled implied volatilities on 20150520

Discrepancy 2

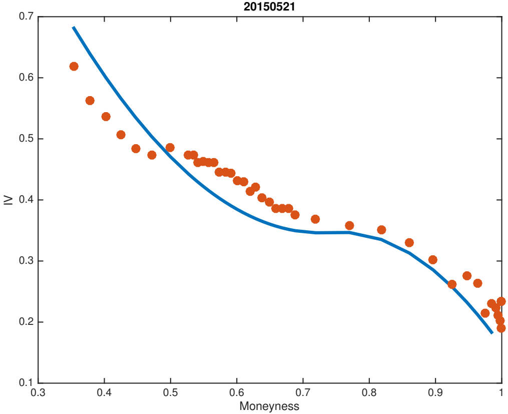


Figure 19: True SSO and MS-DSFM-modeled implied volatilities on 20150521

Forecast evaluation

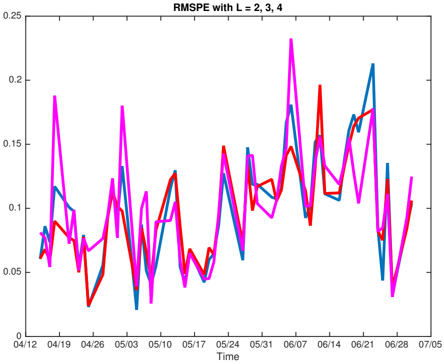


Figure 20: *RMSPE for* $L = 2$, $L = 3$, $L = 4$

Root-mean-square-prediction-error (RMSPE)

$$RMSPE \stackrel{\text{def}}{=} \sqrt{\frac{1}{J_{t+1}} \sum_{j=1}^{J_{t+1}} \left\{ Y_{t+1,j} - \sum_{l=0}^{L} \widehat{Z}_{t+1,l} \widehat{m}_l(X_{t+1,j}) \right\}^2} \qquad (18)$$

Financial gains

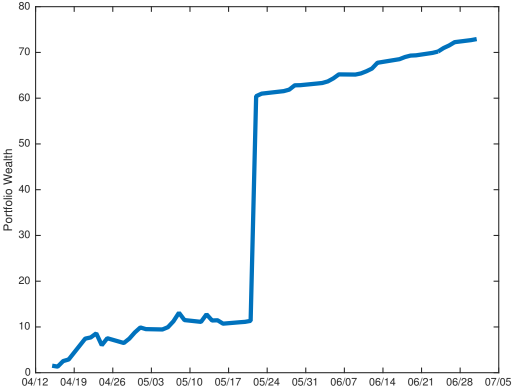


Figure 21: Option portfolios' cumulative gains in 2015; $w = 100$

Stochastic volatility

The main task is to estimate

$$\mathsf{E}^{\mathsf{Q}}\left(\int_0^t \sigma_s^2 \mathrm{d}s \,\middle|\, \log\left(\frac{X_t}{X_0}\right) = LM^{(1)}\right) \tag{19}$$

- different approximations can be made: e.g., Monte-Carlo, series expansions
- what is the error between "true" and approximate expectation?

Stochastic volatility

van der Stoep et al. (2014) study the local stochastic Heston model

$$dX_t = rX_t dt + \sigma(t, X_t)\psi(V_t)dW_{X,t}, \tag{20}$$

$$dV_t = a(t, V_t)dt + b(t, V_t)dW_{V,t}, \tag{21}$$

and estimate

$$g_H \stackrel{\text{def}}{=} \mathsf{E}^{\mathsf{Q}}\left(\psi^2(V_t)|X_t = x\right) \tag{22}$$

by a Monte-Carlo method on a grid of m values:
$x_1 = b_1 \leq b_2 < \ldots < b_m = x_N$

Stochastic volatility

Proposition (van der Stoep et al. (2014))

For an arbitrary bin $\mathcal{B}$ with boundaries $[b_l, b_r]$, the error between g_H and $\widehat{g}_H$ is of the size

$$\|g_H - \widehat{g}_H\|^2_{L^2(\mathcal{B})} = c_1^2 \Delta s + \frac{1}{12}\left(c_2^2 - 2c_1 g_H^{(2)}\right)\Delta s^3 + \frac{1}{240}\left(2(g_H^{(2)}(s_m))^2 - c_1 g_H^{(4)}(s_m) - c_2 g_H^{(3)}(s_m)\right)\Delta s^5 + \mathcal{O}(\Delta s^7), \quad (23)$$

where s_m is the midpoint of $[b_l, b_r]$, $\Delta s \stackrel{\text{def}}{=} b_r - b_l$,
$c_1 \stackrel{\text{def}}{=} 0.5\left\{(g_H(b_l) - \widehat{g}_H(b_l)) + (g_H(b_r) - \widehat{g}_H(b_r))\right\}$,
$c_2 \stackrel{\text{def}}{=} -g_H^{(1)}(s_m) + \frac{\Delta \widehat{g}_H}{\Delta s}$, $\Delta \widehat{g}_H \stackrel{\text{def}}{=} \widehat{g}_H(b_r) - \widehat{g}_H(b_l)$

Monte-Carlo algorithm

1. Generate N pairs of observations (x_i, v_i), $i = 1, \ldots, N$.
2. Order the realizations x_i: $x_1 \leq x_2 \leq \ldots \leq x_N$.
3. Determine the boundaries of M bins $(l_k, l_{k+1}]$, $k = 1, \ldots, M$ on an equidistant grid of values $X^* \stackrel{\text{def}}{=} X_0 e^{LM^{(1)}}$
4. For the kth bin approximate the conditional expectation (6) by

$$\mathsf{E}^{\mathsf{Q}} \left(\int_0^T \sigma_s^2 ds \,|X_T \in (l_k, l_{k+1}] \right) \approx \frac{h}{NQ(k)} \sum_{i=1}^{H} \sum_{j \in \mathcal{J}_k} V_{ij}, \quad (24)$$

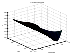
where h is the discretization step for V_t, $\mathcal{J}_k$ the set of numbers j, for which the observations X_T are in the kth bin and $Q(k)$ is the probability of X_T being in the kth bin.

"True" expectation

Proposition

Under the Heston model with risk-neutral dynamics ▸ Details and the volatility risk premium $\lambda = 0$ it holds:

$$\mathrm{E}^{\mathbb{Q}}\left(\int_0^T \sigma_s^2 \mathrm{d}s \,\middle|\, \log\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right) = \left[\int_0^T \frac{\begin{array}{c}\int_0^\infty \Re\left\{\frac{X_0^{-i\phi} e^{-i\phi LM^{(1)}}}{i\phi} f_1(\phi)\left(\frac{\partial C_1}{\partial t} + \frac{\partial D_1}{\partial t} V_0\right)\right. \\ \left. - f_2(\phi)\left(\frac{\partial C_2}{\partial t} + \frac{\partial D_2}{\partial t} V_0\right)\right\} \mathrm{d}\phi\end{array}}{\begin{array}{c}\int_0^\infty \Re\left\{X_0^{-(i\phi+2)} e^{-LM^{(1)}(i\phi+2)}\right. \\ \left. \times f_1(\phi) + X_0 e^{LM^{(1)}} (i\phi - 1) f_2(\phi)\right\} \mathrm{d}\phi\end{array}} dt\right] \times \left(\frac{2}{X_0^2 e^{2LM}}\right), \quad (25)$$



"True" expectation

Proposition (ctd.)

where $\Re$ denotes the real part of a number, assuming $r = q = 0$,

$$f_{1,2}(\phi) = \exp(C_{1,2} + D_{1,2} V_0 + i\phi X_t),$$

$$C_{1,2} = ri\phi\tau + \frac{\kappa\theta}{\sigma^2}\left\{(b_{1,2} - \rho\sigma i\phi - d_{1,2})\tau - 2\log\left(\frac{1 - c_{1,2}e^{-d_{1,2}\tau}}{1 - c_{1,2}}\right)\right\},$$

$$D_{1,2} = \frac{(b_{1,2} - \rho\sigma i\phi - d_{1,2})}{\sigma^2}\left(\frac{1 - e^{-d_{1,2}\tau}}{1 - c_{1,2}e^{-d_{1,2}\tau}}\right),$$

$$\frac{\partial C_{1,2}}{\partial t} = ri\phi + \frac{\kappa\theta}{\sigma^2}\left\{(b_{1,2} - \rho\sigma i\phi + d_{1,2}) + 2\left(\frac{g_{1,2}d_{1,2}e^{d_{1,2}t}}{1 - g_{1,2}e^{d_{1,2}t}}\right)\right\},$$

$$\frac{\partial D_{1,2}}{\partial t} = \frac{(b_{1,2} - \rho\sigma i\phi + d_{1,2})}{\sigma^2}\left(\frac{d_{1,2}e^{d_{1,2}t}(g_{1,2}e^{d_{1,2}t)-1} + (1 - e^{d_{1,2}t})g_{1,2}d_{1,2}e^{d_{1,2}t}}{(1 - g_{1,2}e^{d_{1,2}t})^2}\right),$$

$$d_{1,2} = \sqrt{(\rho\sigma i\phi - b_{1,2})^2 - \sigma^2(2u_{1,2}i\phi - \phi^2)}, \quad c_{1,2} = \frac{b_{1,2} - \rho\sigma i\phi - d_{1,2}}{b_{1,2} - \rho\sigma i\phi + d_{1,2}}$$

$$b_1 = \kappa - \rho\sigma, \quad b_2 = \kappa, \quad u_1 = 0.5, \quad u_2 = -0.5$$

▸ Details

Estimated expectation

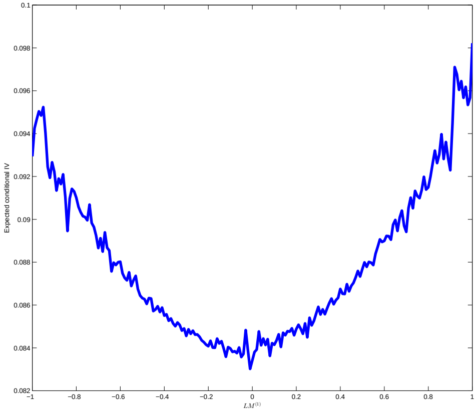


Figure 22: *Estimated value of* $\mathsf{E}^{\mathbb{Q}}(\int_0^t \sigma_s^2 ds | \log(L_t/L_0) = LM^{(1)})$

Estimated expectation: smoothed

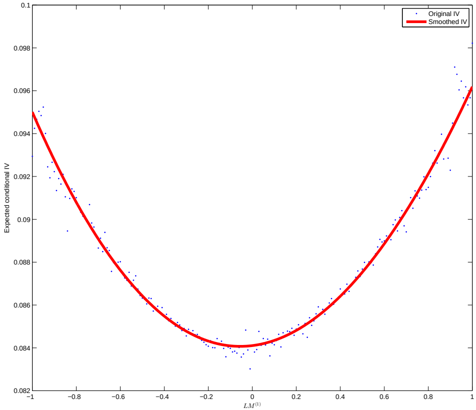


Figure 23: *Smoothed estimate of* $\mathrm{E}^{\mathrm{Q}}(\int_0^t \sigma_s^2 ds | \log(L_t/L_0) = LM^{(1)})$

Conclusions

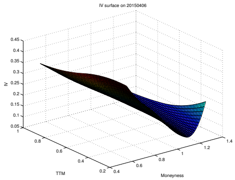
- moneyness scaling effect is not sustainable
- one can use use MS to detect arbitrage opportunities on the LETF option market
- the combined MS and DSFM approach allows to build profitable trading strategies on the LETF option market
- conditional Monte-Carlo or direct computation allow to estimate the "integrated variance smile"

Leveraged ETF options' implied volatility paradox: a statistical study

Wolfgang Karl Härdle

Zhiwu Hong

Sergey Nasekin



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S&P 500 and (L)ETFs Return

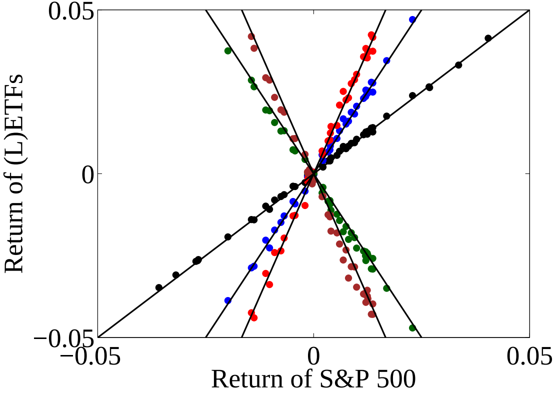

Figure 24: Weekly return relationship of S&P 500 and (L)ETFs (SPY, SSO, UPRO, SDS, SPXU) (20140101-20141230) Back

Moneyness scaling

Given the general solution of (4):

$$L_T = L_t \exp\left\{(r-c)(T-t) - \frac{\beta^2}{2}\int_t^T \sigma_s^2 ds + \beta \int_t^T \sigma_s dW_s^*\right\}, \tag{26}$$

write (26) for $L_T^{(\beta_1)}$, $L_T^{(\beta_2)}$, obtain

$$\frac{L_T^{(\beta_1)}}{e^{(r-c)\tau} L_t^{(\beta_1)}} = \exp\left(-\frac{\beta_1^2}{2}\int_0^\tau \sigma_s^2 ds + \beta_1 \int_0^\tau \sigma_s dW_s\right) \tag{27}$$

$$\frac{L_T^{(\beta_2)}}{e^{(r-c)\tau} L_t^{(\beta_2)}} = \exp\left(-\frac{\beta_2^2}{2}\int_0^\tau \sigma_s^2 ds + \beta_2 \int_0^\tau \sigma_s dW_s\right) \tag{28}$$

where σ_s is the instantaneous volatility at time s.

Moneyness scaling

From (28) follows:

$$\int_t^T \sigma_s dW_s^* = \frac{\log\left(\frac{L_T^{(\beta_2)}}{e^{(r-c)\tau} L_t^{(\beta_2)}}\right) + \frac{\beta_2^2}{2}\int_0^\tau \sigma_s^2 ds}{\beta_2} \tag{29}$$

Substitute (29) into (27) to eliminate the stochastic term $\int_t^T \sigma_s dW_s^*$, obtain:

$$\frac{L_T^{(\beta_1)}}{e^{(r-c)\tau} L_t^{(\beta_1)}} = \exp\left\{-\frac{\beta_1}{2}(\beta_1 - \beta_2)\int_0^\tau \sigma_s^2 \mathrm{d}s\right\}\left\{\frac{L_T^{(\beta_2)}}{e^{(r-c)\tau} L_t^{(\beta_2)}}\right\}^{\frac{\beta_1}{\beta_2}} \tag{30}$$

Moneyness scaling

Take logs and expectations conditioned on $K^{(\beta_1)} = L_T^{(\beta_1)}$ and $K^{(\beta_2)} = L_T^{(\beta_2)}$, obtain

$$\log({k_f}^{(\beta_1)}) = -\frac{\beta_1}{2}(\beta_1 - \beta_2)\mathsf{E}^{\mathsf{Q}}\left(\int_0^T \sigma_s^2 ds \,\middle|\, K^{(\beta_1)} = L_T^{(\beta_1)}, K^{(\beta_2)} = L_T^{(\beta_2)}\right) + \frac{\beta_1}{\beta_2}\log({k_f}^{(\beta_2)})$$

Assuming constant σ and exponentiating, one obtains (5)

▸ Return to "Moneyness scaling"

Tensor product B-splines

Define $U \stackrel{\text{def}}{=} \{\sum_i \alpha_i N_{i,h,s} : \alpha_i \in \mathbb{R}, i \in \mathbb{Z}\}$, $V \stackrel{\text{def}}{=} \left\{\sum_j \beta_j N_{j,k,t} : \beta_j \in \mathbb{R}, j \in \mathbb{Z}\right\}$, then the tensor product B-spline is $b \stackrel{\text{def}}{=} \sum_{i,j} \gamma_{i,j} N_{i,j}$, $\gamma_{i,j} \in \mathbb{R}$, $w \in U \otimes V$, where

$$N_{i,j}(x,y) \stackrel{\text{def}}{=} N_{i,h,s}(x) N_{j,k,t}(y),$$

$$N_{i,k,t}(x) \stackrel{\text{def}}{=} \left(\frac{x - t_i}{t_{i+k} - t_i}\right) N_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) N_{i+1}^{k-1}(x),$$

with the starting point

$$N_{i,0,t}(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } t_i \le x < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

here $k \in \mathbb{N}$, t_i infinite set of knots Back to "DSFM for IVS"

Tensor B-spline derivatives

- a B-spline surface $b(x, y)$ can be represented in Bézier form, Prautzsch et al. (2002)

$$b(x, y) = \sum_i \sum_j \beta_{ij} B_i^n(x) B_j^k(y), \tag{31}$$

where B_i^n are Bernstein polynomials ▸ Details

- the partial derivatives of (31) are given by

$$\frac{\partial^{q+r} b(x, y)}{\partial x^q \partial y^r} = \frac{n!k!}{n!k! - qr} \sum_i \sum_j \Delta^{01} \Delta^{q,r-1} \beta_{ij} B_i^{n-q}(x) B_j^{k-r}(y), \tag{32}$$

where the forward difference $\Delta^{qr} \beta_{ij} = \beta_{i+q,j+r} - \beta_{i,j}$

▸ Return to "Tensor B-spline derivatives"

Bernstein polynomials

Bernstein polynomials of degree n are given by

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i},$$

where $i = 0, \ldots, n$ Return to "Tensor B-spline derivatives"

Convergence conditions

Initial choice (α^0, Z^0) such that, Park et al. (2009):

A1 it holds that $\sum_{i=1}^{I} Z_i^0 = 0$; $\sum_{i=1}^{I} Z_i^0 Z_i^{0\top}$ and the Hessian from (12) at (α^0, Z^0), $\mathcal{H}(\alpha^0, Z^0)$ are invertible

A2 there exists a version $(\widehat{\alpha}, \widehat{Z})$ with $\sum_{i=1}^{I} \widehat{Z}_i = 0$ such that $\sum_{i=1}^{I} \widehat{Z}_i Z_i^{0\top}$ is invertible. Also, $\hat{\alpha}_l = (\hat{\alpha}_{l1}, \ldots, \hat{\alpha}_{lK})^\top$, $l = 0, \ldots, L$ are linearly independent

Return to "Identification"

Numeric algorithm I

The first-order conditions for (12):

$$\frac{\partial S(\mathcal{A}, Z)}{\partial \alpha} = 2\sum_{i=1}^{I} \left\{ (\Psi_i \Psi_i^\top) \otimes (\mathcal{Z}_i \mathcal{Z}_i^\top) \right\} \alpha - 2\sum_{i=1}^{I} (\Psi_i Y_i) \otimes \mathcal{Z}_i, \tag{33}$$

$$\frac{\partial S(\mathcal{A}, Z)}{\partial Z} = 2(\mathcal{Z}_1^\top \mathcal{A} \Psi_1 \Psi_1^\top A^\top - Y_1^\top \Psi_1^\top A^\top, \ldots, \mathcal{Z}_I^\top \mathcal{A} \Psi_I \Psi_I^\top A^\top - Y_I^\top \Psi_I^\top A^\top), \tag{34}$$

where A is $\mathcal{A}$ without 1st row, $\Psi_i \stackrel{\text{def}}{=} \{\psi(X_{i,1}), \ldots, \psi(X_{i,J_i})\}$, $\alpha \stackrel{\text{def}}{=} \text{vec}(\mathcal{A})$ Return to "Identification"

Numeric algorithm II

The second-order conditions for (12):

$$\frac{\partial^2 S(\mathcal{A}, Z)}{\partial \alpha^2} = 2\sum_{i=1}^{I}\left\{(\Psi_i\Psi_i^\top)\otimes(\mathcal{Z}_i\mathcal{Z}_i^\top)\right\}, \tag{35}$$

$$\frac{\partial^2 S(\mathcal{A}, Z)}{\partial Z^2} = \begin{pmatrix} A\Psi_1\Psi_1^\top A^\top & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A\Psi_I\Psi_I^\top A^\top \end{pmatrix}, \tag{36}$$

$$\frac{\partial^2 S(\mathcal{A}, Z)}{\partial \alpha \partial Z} = 2\{F_1(\alpha, Z), \ldots, F_I(\alpha, Z)\}, \tag{37}$$

where

$F_i(\alpha, Z) \overset{\text{def}}{=} (\Psi_i\Psi_i^\top A^\top)\otimes\mathcal{Z}_i + (\Psi_i\Psi_i^\top \mathcal{A}^\top\mathcal{Z}_i)\otimes\mathcal{I} - (\Psi_i Y_i)\otimes\mathcal{I}$,

$\mathcal{I} = (0, I_L)$, I_L is $L\times L$ identity matrix ▸ Return to "Identification"

Numeric algorithm III

Collect the FOCs (33)-(34) and the SOCs (35)-(37) into the Newton iteration for (12):

$$x_{k+1} = x_k - \mathcal{H}^{-1}(x_k)\nabla(x_k), \tag{38}$$

where $x_k \stackrel{\text{def}}{=} \begin{pmatrix} \alpha^{(k)} \\ Z^{(k)} \end{pmatrix}$, $\mathcal{H}^{-1}(x_k) \stackrel{\text{def}}{=} \left.\begin{pmatrix} \frac{\partial^2 S(\mathcal{A},Z)}{\partial \alpha^2} & \frac{\partial^2 S(\mathcal{A},Z)}{\partial \alpha \partial Z} \\ \frac{\partial^2 S(\mathcal{A},Z)}{\partial \alpha \partial Z}^\top & \frac{\partial^2 S(\mathcal{A},Z)}{\partial Z^2} \end{pmatrix}\right|_{x_k}$,

$\nabla(x_k) \stackrel{\text{def}}{=} \left.\begin{pmatrix} \frac{\partial S(\mathcal{A},Z)}{\partial \alpha} \\ \frac{\partial S(\mathcal{A},Z)}{\partial Z} \end{pmatrix}\right|_{x_k}$

Return to "Identification"

Stable vector autoregressive process

VAR(p) process

$$y_t = c + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t, \quad t \in \mathbb{Z}, \tag{39}$$

where $y_t \in \mathbb{R}^k$ random vector, $A_i \in \mathbb{R}^{k \times k}$ fixed coefficient matrices, $c \in \mathbb{R}^k$ fixed vector of intercept terms, $u_t \in \mathbb{R}^k$ innovation process, $\operatorname{E} u_t = 0$, $\operatorname{E} u_t u_s^\top = 0$, $s \neq t$, $\Sigma_u \stackrel{\text{def}}{=} \operatorname{E} u_t u_t^\top$ is called *stable* if

$$\det(I_k - A_1 z - \cdots - A_p z^p) \neq 0 \quad \text{for} \quad |z| \leq 1,$$

i.e., the reverse characteristic polynomial of (39) has no roots inside and on the complex unit circle Return to "Estimation"

Heston model

Bivariate system of stochastic differential equations, see Heston (1993):

$$dX_t = (r - q - 0.5)X_t dt + \sqrt{V_t} X_t dW_{X,t}^{\mathbb{Q}}, \tag{40}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{V,t}^{\mathbb{Q}}, \tag{41}$$

where $dW_{X,t}$, $dW_{V,t}$ are Wiener processes correlated with parameter ρ Return to "True" expectation

Proof sketch

The Dupire formula allows to compute the local volatility of a European option, defined by

$$\sigma^2_{K,T}(X_t, t) \overset{\text{def}}{=} \mathsf{E}^{\mathsf{Q}}\{\sigma^2(X_T, T, t, \cdot)|X_T = K, \mathcal{F}_t\} \tag{42}$$

$$= \frac{\frac{\partial C}{\partial T}}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}} \tag{43}$$

▸ Return to "True" expectation

Proof sketch

Given the Heston setup, $\sigma^2(X_T, T, t, \cdot) = V_t$, some constant $LM^{(1)}$, it follows that

$$
\begin{aligned}
\mathrm{E}^{\mathrm{Q}}\left(\int_0^T V_t \mathrm{d}t \,\middle|\, \log\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right) &= \int_0^T \mathrm{E}^{\mathrm{Q}}\left(V_t \,\middle|\, \log\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right) \mathrm{d}t \\
&= \int_0^T \mathrm{E}^{\mathrm{Q}}\left(V_t \,\middle|\, X_T = X_0 \exp(LM^{(1)})\right) \mathrm{d}t \\
&= \int_0^T \mathrm{E}^{\mathrm{Q}}\left(V_t \,\middle|\, X_T = \widetilde{K}\right) \mathrm{d}t,
\end{aligned}
$$

where $\widetilde{K}$ is some constant strike price for a european call option on the ETF X.

▸ Return to "True" expectation

Proof sketch

Applying the Dupire formula (43) to $\mathrm{E}^{\mathrm{Q}}\left(V_t \middle| X_T = \widetilde{K}\right)$ yields

$$\mathrm{E}^{\mathrm{Q}}\left(\int_0^T V_t \mathrm{d}t \middle| \log\left(\frac{X_T}{X_0}\right) = LM^{(1)}\right) = \int_0^T \mathrm{E}^{\mathrm{Q}}\left(V_t \middle| X_T = \widetilde{K}\right)\mathrm{d}t$$

$$= \int_0^T \frac{\frac{\partial C_H}{\partial T}}{\frac{\widetilde{K}^2}{2}\frac{\partial^2 C_H}{\partial \widetilde{K}^2}}\mathrm{d}t,$$

where C_H is the Heston price of the call option. The partial derivatives $\partial C_H/\partial T$ and $\partial^2 C_H/\partial \widetilde{K}^2$ are the "Heston Greeks" and are given, e.g., in Rouah (2013). After simplifications, one obtains (25)

▸ Return to "True" expectation